# Response of Periodic Structures by the Z-Transform Method

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**Periodic structures are defined as structures consisting of identical substructures connected to each other in identical manner. The response of periodic structures to harmonic excitation can be described by a matrix difference equation. The solution of the matrix difference equation can be obtained by the Z-transform method and it yields the response to both end conditions and external excitations. The method developed necessitates the eigenvalues of the transfer matrix for a typical substructure, so that the procedure is capable of analyzing a periodic structure with the same computational effort necessary to analyze a single substructure. Added advantage is derived from the fact that the method does not require the eigenvectors of the transfer matrix.**

## Introduction

A PERIODIC structure is a structure consisting of identical substructures connected to each other in an identical manner. Many physical systems in natural state possess properties that are periodic in space. A typical example is the single crystal in which identically arranged atoms form an infinite or semi-infinite lattice. Many other physical systems are built in the form of periodic structures by design, the object being to reduce cost or to save time or both. For example, a segment of an aircraft fuselage is often made of identical bays connected by identical circumferential frames. Even the circumferential frames in an aircraft fuselage can be looked upon as periodic structures. Monorail tracks, or pontoon bridges can be regarded as examples of periodic structures. Continuous systems with periodic properties can often be treated as discrete periodic systems. The mathematical formulation for discrete periodic systems can be effected by means of the finite element method. The formulation consists of a set of simultaneous ordinary differential equations.

There is no single approach to the mathematical analysis of periodic structures. Indeed, the approach depends very much on the nature of the substructure. In particular, it is possible to distinguish between a scalar approach and a matrix approach to the problem formulation. This is related directly to the order of the substructure. More specifically, the scalar approach is suitable for the case in which the substructure represents a single-degree-of-freedom system, whereas the matrix approach is recommended for the case in which the substructure represents a multi-degree-of-freedom system. The scalar analysis of periodic structures is in a relatively advanced stage of development. Much progress has been made in the matrix analysis of periodic structures but much progress remains to be made. In the following we present a survey of selected works on the subject.

A classical example of a simple periodic structure is the one-dimensional array of identical linear springs. Hence, the substructure is the simple harmonic oscillator. Using such a mathematical model, Brillouin[1] has studied the propagation of harmonic waves in discrete crystal lattices. The same problem is discussed in the text by Morse and Ingard[2] who present solutions to both sinusoidal wave motion and transient motion in infinite arrays, where in the latter case the dispersive character of the array for wave motion is pointed out. In a related work, Weinstock[3] derives the response of a semi-infinite lattice (of the type treated in Refs. 1 and 2) to an excitation in the form of a uniform velocity imparted to the end particle. He proposes a method whereby the infinite set of ordinary differential equations is replaced by a single differential equation. Addressing himself to the same problem as that of Ref. 3, Goodman[4] presents a solution in a slightly more general form, obtained by a procedure he refers to as the "response-function method." In an attempt to gain some insight into the behavior of elastic and viscoelastic composite materials, Nayfeh[5] investigates the same mathematical model as that of Refs. 1-4, but in addition he considers viscous damping both in series and in parallel with the springs. Exact and approximate solutions are obtained by integral transform techniques.

References 1-5 are concerned with periodic structures whose basic unit is a single-degree-of-freedom system. They all adopt essentially a scalar approach, in which the dependent variable is the displacement. The same problem can be formulated in terms of a two-dimensional "spatial state vector" whose components are the displacement and the force at every mass point. If the time is eliminated from the formulation, either by assuming harmonic motion or by means of the Laplace transform, then the original differential equations reduce to difference equations. As a result, the relation between the state at one particle and the state at an adjacent particle can be written in terms of a $2 \times 2$ transfer matrix. This gives rise to a formulation that has come to be known as "matrix difference equation." Lin and McDaniel[6] investigate the matrix difference equation associated with a periodic Bernoulli-Euler beam on many elastic supports, where the beam is of finite length. They point out various numerical difficulties that may be encountered in using a conventional transfer matrix approach, involving the multiplication of a chain of transfer matrices, and propose a "complementary approach" to circumvent these numerical difficulties. A generalization of the matrix difference technique is provided by Mead,[7] who considers the problem of harmonic wave propagation in arbitrary one-dimensional periodic arrays with multiple coupling. He shows that the frequencies at which wave motion can occur for any given propagation constant can be obtained by solving an eigenvalue problem. An approach similar to that of Ref. 7, but more efficient from a computational point of view, is presented by Denke et al.[8] They properly identify the objective of an efficient analysis of a periodic structure, namely, to render the computing effort independent of the number of substructures. Consistent with this idea, they give the solution of the difference equations in terms of eigenvalues and eigenvectors of a matrix derived from the substructure mechanical impedance matrix.

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Although concerned with the same type of mathematical models as those of Refs. 7 and 8, the approach to the mathematical solution proposed here is distinctly different. Moreover, the approach can accommodate not only end excitations but also external forces at the boundaries interconnecting the various substructures. In addition, it can accommodate semi-infinite as well as finite periodic structures. The solution of the matrix difference equation is obtained by the $Z$-transform method, which yields the response to both end conditions and distributed external excitations. The procedure takes advantage of certain properties of transfer matrices to develop an efficient computational algorithm for inverting $Z$-transforms. To this end, it is only necessary to obtain the eigenvalues (but not the eigenvectors) of the transfer matrix for a typical substructure, so that the procedure is capable of analyzing a periodic structure with the same computational effort necessary to analyze a single substructure.

## The Matrix Difference Equation

Let us consider a discrete system consisting of a set of subsystems $i$ and denote by $\boldsymbol{u}_{i,L}$ and $\boldsymbol{u}_{i,R}$ the displacement vectors on the left and on the right side of subsystem $i$(Fig. 1). Similarly, $\boldsymbol{p}_{i,L}+\boldsymbol{f}_{i,L}$ and $\boldsymbol{p}_{i,R}+\boldsymbol{f}_{i,R}$ are force vectors, where $\boldsymbol{p}$ denotes internal forces and $\boldsymbol{f}$ external forces. Then, considering the case in which the forces and the displacements are harmonic, oscillating with the frequency $\omega$, and using a procedure similar to that of Ref. 8, we obtain the *matrix difference equation*

$$\boldsymbol{x}_{i+1,L}=A_i\boldsymbol{x}_{i,L}+\boldsymbol{f}_i^* \tag{1}$$

where

$$\boldsymbol{x}_{i,L}=\begin{bmatrix}\boldsymbol{u}_{i,L}\\ \hline \boldsymbol{p}_{i,L}+\boldsymbol{f}_{i,L}\end{bmatrix} \tag{2}$$

is a "spatial state vector" describing the state of the system on the left side of substructure $i$,

$$A_i=\left[\begin{array}{c|c}z_{i,LR} & 0\\ \hline z_{i,RR} & 1\end{array}\right]^{-1}\left[\begin{array}{c|c}-z_{i,LL} & 1\\ \hline -z_{i,RL} & 0\end{array}\right]$$

$$=\left[\begin{array}{c|c}-z_{i,LR}^{-1}z_{i,LL} & z_{i,LR}^{-1}\\ \hline z_{i,RR}z_{i,LR}^{-1}z_{i,LL}-z_{i,RL} & -z_{i,RR}z_{i,LR}^{-1}\end{array}\right] \tag{3}$$

is a *transfer matrix* relating the states $\boldsymbol{x}_{i,L}$ and $\boldsymbol{x}_{i+1,L}$ and

$$\boldsymbol{f}_i^*=\begin{bmatrix}0\\ \hline \boldsymbol{f}_{i,R}+\boldsymbol{f}_{i+1,L}\end{bmatrix} \tag{4}$$

is the *net external force vector* acting between the subsystems $i$ and $i+1$. Note that $z_{i,LL},\ldots$ are submatrices of the impedance matrix $Z_i$ for the subsystem $i$, where $Z_i$ is a symmetric matrix.

Matrix $A_i$ depends in general on the index $i$. In this case a strictly numerical solution of Eq. (1) can be obtained by means of a digital computer. If the system is periodic in space, i.e., if the subsystems are identical, then $A_i=A=\text{const}$ and a more formal solution of Eq. (1) is possible.

## Response of a Periodic Structure

Let us assume that the system is periodic in space, so that Eq. (1) reduces to

$$\boldsymbol{x}_{i+1}=A\boldsymbol{x}_i+\boldsymbol{f}_i^* \tag{5}$$

where the subscript $L$ has been dropped, with the understanding that the state vector $\boldsymbol{x}_i$ is evaluated at the left boundary of a substructure. Equation (5) represents a matrix difference equation with constant coefficients.

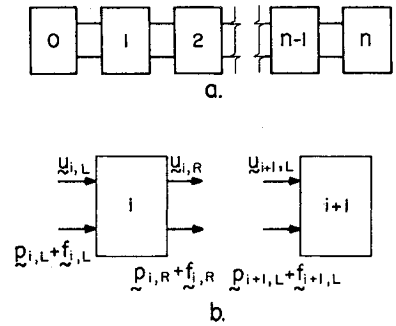


**Fig. 1 The periodic system.**

The response of periodic structures can be conveniently obtained by means of the $Z$-transform. To this end, let us introduce the $Z$-transform of $\boldsymbol{x}_i$ defined as (see Ref. 9)

$$\boldsymbol{x}(z)=Z[\boldsymbol{x}_i]=\sum_{i=0}^{\infty}\boldsymbol{x}_i z^{-i} \tag{6}$$

and note that, although the upper limit of the sum is infinity, the approach is valid not only for semi-infinite structures but also for finite ones (see Ref. 9, p. 46). Similarly, let us define

$$\boldsymbol{f}^*(z)=Z[\boldsymbol{f}_i^*] \tag{7}$$

Moreover, it can be easily verified that

$$Z[\boldsymbol{x}_{i+1}]=z\boldsymbol{x}(z)-z\boldsymbol{x}_0 \tag{8}$$

where $\boldsymbol{x}_0$ is the state vector on the left side of the substructure 0. Taking the $Z$-transform on both sides of Eq. (5) and rearranging, we obtain the transformed response

$$\boldsymbol{x}(z)=z[zI-A]^{-1}\boldsymbol{x}_0+[zI-A]^{-1}\boldsymbol{f}^*(z) \tag{9}$$

The actual response $\boldsymbol{x}_i$ can be determined from Eq. (9), provided the inverse $Z$-transform of $\boldsymbol{x}(z)$ can be obtained. This, in turn, requires that $\boldsymbol{x}_0$ and $\boldsymbol{f}^*(z)$ be known.

Next, let us introduce

$$Z^{-1}\{z[zI-A]^{-1}\}=\Phi_i \tag{10}$$

from which it follows that

$$Z^{-1}[zI-A]^{-1}=\Phi_{i-1} \tag{11}$$

Considering Eqs. (10) and (11) and using the convolution theorem for $Z$-transforms, we obtain the response

$$\boldsymbol{x}_i=\Phi_i\boldsymbol{x}_0+\sum_{k=0}^{i-1}\Phi_{i-k-1}\boldsymbol{f}_k^* \tag{12}$$

The matrix $\Phi_i$ is known as the *fundamental matrix* for the system, so that the response problem reduces to the determination of the fundamental matrix. Using Eq. (5) recursively by letting $i=0,1,2,\ldots$, it can be shown that $\Phi_i=A^i$. As $i$ becomes large, however, accuracy is lost in evaluating $A^i$, so that a method whose accuracy does not depend on $i$ appears highly desirable.

In using Eq. (10) to produce the fundamental matrix $\Phi_i$, it is necessary to compute the inverse of the matrix $[zI-A]$, which is a matrix polynomial in $z$. The problem becomes progressively more difficult as the order of the subsystem increases, so that an efficient method for obtaining the inverse of $[zI-A]$ for higher order systems is highly desirable. Such a method is the *Leverrier algorithm*,[10] according to which the inverse of the matrix $[zI - A]$ can be written in the form

$$[zI-A]^{-1} = \frac{1}{\det [zI-A]} \sum_{\ell=0}^{2n-1} z^{2n-\ell-1} H_\ell \qquad (13)$$

where

$$\det [zI-A] = z^{2n} + \theta_1 z^{2n-1} + \theta_2 z^{2n-2} + \ldots + \theta_{2n} \qquad (14)$$

is the characteristic polynomial of the matrix $A$ and

$$H_0 = I, \quad H_\ell = AH_{\ell-1} + \theta_\ell I, \quad \ell = 1,2,\ldots,2n-1 \qquad (15)$$

are constant square matrices of order $2n$. Moreover, the coefficients $\theta_\ell$ of the characteristic polynomial are given by

$$\theta_\ell = -\frac{1}{\ell} \operatorname{tr} AH_{\ell-1}, \quad \ell = 1,2,\ldots,2n \qquad (16)$$

Using Eqs. (14) and (15) in conjunction with the Cayley-Hamilton theorem, it can be easily verified that

$$H_{2n} = AH_{2n-1} + \theta_{2n} I = 0 \qquad (17)$$

which can be used to check the method. Note that $n$ is the number of degrees of freedom of the subsystem.

Next, let us introduce the notation

$$X(z) = [zI-A]^{-1} = \frac{1}{\det [zI-A]} \sum_{\ell=0}^{2n-1} z^{2n-\ell-1} H_\ell \qquad (18)$$

and consider a partial fractions expansion of the matrix $X(z)$. The matrix has poles at the zeros of the characteristic polynomial of $A$. We shall denote these zeros by $z_j$ ($j = 1,2,\ldots,2n$), so that, assuming that these zeros are all distinct, the partial fractions expansion of $X(z)$ can be written in the form

$$X(z) = \sum_{j=1}^{2n} \frac{1}{z - z_j} X_j \qquad (19)$$

where the constant coefficient matrices have the expressions

$$X_j = \lim_{z \to z_j} [(z-z_j)X(z)] = \frac{\sum_{\ell=0}^{2n-1} z_j^{2n-\ell-1} H_\ell}{\sum_{\ell=0}^{2n-1} (2n-\ell)\theta_\ell z_j^{2n-\ell-1}}$$

$$j = 1,2,\ldots,2n \qquad (20)$$

in which $\theta_0 = 1$. Hence, using Eq. (20), the fundamental matrix becomes

$$\Phi_i = Z^{-1}[zX(z)] = Z^{-1}\left[\sum_{j=1}^{2n} \frac{z}{z-z_j} X_j\right] \qquad (21)$$

But, from tables of inverse Z-transforms, it can be easily verified that

$$\Phi_i = \sum_{j=1}^{2n} z_j^i X_j \qquad (22)$$

so that, inserting Eq. (22) into Eq. (12), we obtain the general response

$$x_i = \sum_{j=1}^{2n} X_j \left(z_j^i x_0 + \sum_{k=0}^{i-1} z_j^{i-k-1} f_k^*\right) \qquad (23)$$

The above method requires the determination of the roots of the characteristic polynomial (14), which are really the eigenvalues of the matrix A. This is a considerably easier task than obtaining both the eigenvalues and eigenvectors of $A$, as demanded by the method proposed in Ref. 8. Moreover, it should be pointed out that eigenvalues can be obtained with significantly higher accuracy than eigenvectors, so that the method proposed here is likely to be more accurate.

The roots of the polynomial (14) are in general complex and can be obtained by a variety of techniques. The matrix $A$, however, is a transfer matrix, and it possesses not only interesting but also useful properties. These properties can be used to reduce the computational effort significantly, as shown in the next section.

## Properties of the Matrix $A$. The Eigenvalue Problem

To obtain the system response, it is necessary to calculate the eigenvalues of the matrix $A$. From Eq. (3), we conclude that $A$ can be written in the form of the matrix product

$$A = \begin{bmatrix} z_{LR}^{-1} & 0 \\ -z_{RR}z_{LR}^{-1} & I \end{bmatrix} \begin{bmatrix} -z_{LL} & I \\ -z_{RL} & 0 \end{bmatrix} \qquad (24)$$

where the index $i$ has been dropped because the submatrices in question are the same for every substructure. But the determinant of a product of matrices is equal to the product of the determinants of the matrices, i.e.,

$$\det A = \det \begin{bmatrix} z_{LR}^{-1} & 0 \\ -z_{RR}z_{LR}^{-1} & I \end{bmatrix} \det \begin{bmatrix} -z_{LL} & I \\ -z_{RL} & 0 \end{bmatrix} \qquad (25)$$

It is not difficult to show, however, that

$$\det \begin{bmatrix} z_{LR}^{-1} & 0 \\ -z_{RR}z_{LR}^{-1} & I \end{bmatrix} = \det z_{LR}^{-1} \qquad (26a)$$

$$\det \begin{bmatrix} -z_{LL} & I \\ -z_{RL} & 0 \end{bmatrix} = \det z_{RL} = \det z_{LR} \qquad (26b)$$

where the second equality in Eq. (26b) is true because $z_{LR} = z_{RL}^T$. Inserting Eqs. (26) into Eq. (25), we conclude that

$$\det A = I \qquad (27)$$

from which it follows immediately that

$$\det A^{-1} = I \qquad (28)$$

Next, let us consider the eigenvalue problem associated with $A$. Recalling Eq. (3), we can write the problem in the form

$$\det[\lambda I - A] = \det \left[ \lambda I - \begin{bmatrix} z_{LR} & 0 \\ z_{RR} & I \end{bmatrix}^{-1} \begin{bmatrix} -z_{LL} & I \\ -z_{RL} & 0 \end{bmatrix} \right]$$

$$= \det \begin{bmatrix} \lambda z_{LR} + z_{LL} & -I \\ \lambda z_{RR} + z_{RL} & \lambda I \end{bmatrix} = 0 \qquad (29)$$

Because the value of a determinant does not change if we add one row multiplied by a constant to another row, we can multiply the top half of the determinant by $\lambda$ and add to the bottom half to obtain

$$\det \begin{bmatrix} \lambda z_{LR} + z_{LL} & -I \\ \lambda^2 z_{LR} + \lambda(z_{LL} + z_{RR}) + z_{RL} & 0 \end{bmatrix}$$

$$= \det[\lambda^2 z_{LR} + \lambda(z_{LL} + z_{RR}) + z_{RL}] = 0 \qquad (30)$$

Now let us consider the eigenvalue problem associated with $A^{-1}$, denote the eigenvalues of $A^{-1}$ by $\bar{\lambda}$, and write

$$\det[\bar{\lambda}I - A^{-1}] = \det\left[\bar{\lambda}I - \begin{bmatrix} -z_{LL} & I \\ -z_{RL} & 0 \end{bmatrix}\begin{bmatrix} z_{LR} & 0 \\ z_{RR} & I \end{bmatrix}\right]$$

$$= \det\begin{bmatrix} -\bar{\lambda}z_{LL} - z_{LR} & \bar{\lambda}I \\ -\bar{\lambda}z_{RL} - z_{RR} & -I \end{bmatrix}$$

$$= \det[-\bar{\lambda}^2 z_{RL} - \bar{\lambda}(z_{LL} + z_{RR}) - z_{LR}] = 0 \tag{31}$$

Because $z_{RL} = z_{LR}^T$, $z_{LL} = z_{LL}^T$, $z_{RR} = z_{RR}^T$, we conclude that $\lambda$ and $\bar{\lambda}$ satisfy the same eigenvalue problem. But $\bar{\lambda} = 1/\lambda$, so that *the eigenvalues of* $A$ *( or of* $A^{-1}$*) occur in reciprocal pairs* $\lambda$ *and* $1/\lambda$. This fact has significant implications as far as the computation of the eigenvalues of $A$ is concerned, because we need calculate only one half of the eigenvalues.

Next, let us consider Eq. (14). Letting $z = \lambda$, we can write the characteristic equation associated with a typical substructure in the form

$$\lambda^{2n} + \theta_1\lambda^{2n-1} + \theta_2\lambda^{2n-2} + \ldots + \theta_{2n} = 0 \tag{32}$$

Dividing Eq. (32) through by $\lambda^{2n}$, we obtain

$$\theta_{2n}\bar{\lambda}^{2n} + \theta_{2n-1}\bar{\lambda}^{2n-1} + \theta_{2n-2}\bar{\lambda}^{2n-2} + \ldots + 1 = 0 \tag{33}$$

Because $\lambda$ and $\bar{\lambda}$ satisfy the same characteristics equation, we conclude that

$$\theta_{2n} = 1, \quad \theta_{2n-\ell} = \theta_\ell, \quad \ell = 1,2,\ldots,2n-1 \tag{34}$$

From Eqs. (34), we conclude that we need calculate only one half of the coefficients $\theta_\ell$. Hence, for the case in which the substructure has $n$ degrees of freedom, the characteristic equation can be obtained by merely calculating $n$ coefficients $\theta_\ell$.

In calculating the matrices $H_\ell$, a saving similar to that for the coefficients $\theta_\ell$ can be achieved. To show this, let us consider the matrix $A$ and write it in the general partitioned form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{35}$$

where the $n \times n$ submatrices $a_{11}$, $a_{12}$, $a_{21}$, and $a_{22}$ are as shown in Eq. (3). Considering the special nature of the submatrices in Eq. (3), it is not difficult to prove that the inverse of $A$ is given by

$$A^{-1} = \begin{bmatrix} a_{22}^T & -a_{12}^T \\ -a_{21}^T & a_{11}^T \end{bmatrix} \tag{36}$$

so that the calculation of the inverse of $A$ is a relatively simple process. More importantly, the inversion process (36) does not involve any loss of accuracy because it merely involves transposition, which is in direct contrast with the inversion of ordinary matrices. Equation (36) can be easily explained by the fact that $A$ represents a transfer matrix relating the state vector $x_{i+1,L}$ to $x_{i,L}$, so that $A^{-1}$ represents a transfer matrix in the opposite direction, i.e., one relating $x_{i,L}$ to $x_{i+1,L}$.

Next, let us define the matrix operator $T$ in the form

$$T(B) = \begin{bmatrix} b_{22}^T & -b_{12}^T \\ -b_{21}^T & b_{11}^T \end{bmatrix} \tag{37}$$

where $b_{11},\ldots$ are submatrices of the matrix $B$. Clearly, $T$ exhibits the following properties

$$T(a_1A_1 + a_2A_2) = a_1T(A_1) + a_2T(A_2)$$
$$T(A_1A_2) = T(A_2)T(A_1) \tag{38}$$

Examining Eqs. (15) and using simple substitution, it can be verified that

$$H_\ell = \sum_{k=0}^{\ell} \theta_{\ell-K}A^k \tag{39}$$

so that Eqs. (36) and (38) yield

$$T(AH_\ell) = 3\sum_{k=0}^{\ell} \theta_{\ell-k}A^{-(k+1)} \tag{40}$$

On the other hand, from Eqs. (15), it can also be shown that

$$H_{2n-(\ell+1)} = -\sum_{k=0}^{\ell} \theta_{2n-(\ell-k)}A^{-(k+1)} \tag{41}$$

In view of relations (34), however, we obtain

$$H_{2n-(\ell+1)} = -T(AH_\ell) \tag{42}$$

Hence, it is only necessary to calculate one-half of the matrices $H_\ell$ ($\ell = 1,2,\ldots,2n-1$), as the balance can be obtained by the simple process (37).

The solution outlined above requires the eigenvalues of $A$. These can be obtained by solving the eigenvalue problem of $A$ directly or by solving Eq. (32). In general the eigenvalues $\lambda$ are complex, so that a typical eigenvalue can be written in the form $\lambda = ae^{j\phi}$, where $a$ is the magnitude of $\lambda$ and $\phi$ is the phase angle. (Note that according to the notation used here $j = \sqrt{-1}$. This $j$ should not be confused with that identifying a particular root.) Then, the reciprocal eigenvalue is simply $1/\lambda = (1/a)e^{-j\phi}$. From Eq. (23), we conclude that the solution involves the roots $z_j$ to the power $i$. However, for semi-infinite structures $i \to \infty$, so that the terms corresponding to roots with magnitudes larger than unity diverge. Because these terms must be discarded, the interest lies only in eigenvalues with magnitudes smaller than unity, which narrows the search for the roots substantially. If the periodic structure is finite, however, we must keep both roots $\lambda$ and $1/\lambda$. In this case, it is possible to take advantage of the properties (34) of the coefficients $\theta_\ell$ of the characteristic polynomial to reduce the order of the characteristic by a factor of two. Such reductions

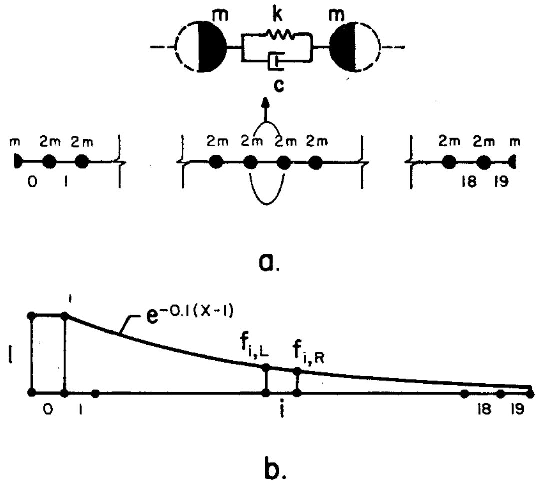


**Fig. 2 a) System in axial vibration. b) External load.**

are particularly desirable for large-order polynomials. A procedure for reducing the order of the polynomial is developed in the Appendix.

It should be pointed out that added efficiency can be obtained when the substructures are symmetric. This aspect will be explored in detail in a future paper.

## Illustrative Example

Let us consider the periodic system in axial vibration shown in Fig. 2a. The substructures can be identified as single-degree-of-freedom damped systems. It is easy to verify that the mass matrix, damping matrix, and stiffness matrix for a typical substructure have the form

$$M=m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ C=c\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix},\ K=k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \tag{43}$$

so that the impedance matrix is simply

$$Z=\begin{bmatrix}-\omega^2 m+j\omega c+k & -(j\omega c+k)\\ -(j\omega c+k) & -\omega^2 m+j\omega c+k\end{bmatrix} \tag{44}$$

Using Eq. (3), we can write the transfer matrix

$$A=\begin{bmatrix}\alpha & -1/\beta\\ \beta(1-\alpha^2) & \alpha\end{bmatrix} \tag{45}$$

where

$$\alpha=\frac{-\omega^2 m+j\omega c+k}{j\omega c+k}\quad ,\quad \beta=j\omega c+k \tag{46}$$

Next, we shall use the Leverrier algorithm to obtain the fundamental matrix $\Phi_i$. To this end, we use Eqs. (15) and (16) and obtain

$$\theta_1=-\text{tr}\ A=-2\alpha$$

$$H_1=A-2\alpha I=\begin{bmatrix}-\alpha & -1/\beta\\ \beta(1-\alpha^2) & -\alpha\end{bmatrix} \tag{47}$$

Hence, Eq. (18) has the form

$$X(z)=\frac{1}{z^2-2\alpha z+1}[zH_0+H_1]$$

$$=\frac{1}{z^2-2\alpha z+1}\begin{bmatrix}z-\alpha & -1/\beta\\ \beta(1-\alpha^2) & z-\alpha\end{bmatrix} \tag{48}$$

The characteristic equation is simply

$$z^2-2\alpha z+1=0 \tag{49}$$

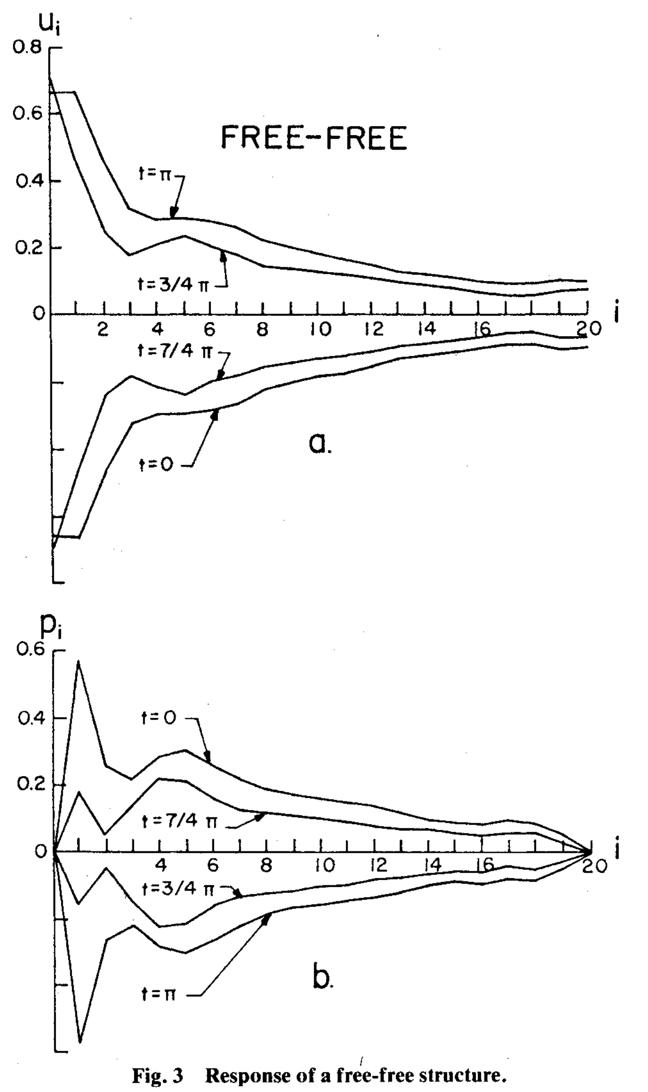


**Fig. 3 Response of a free-free structure.**

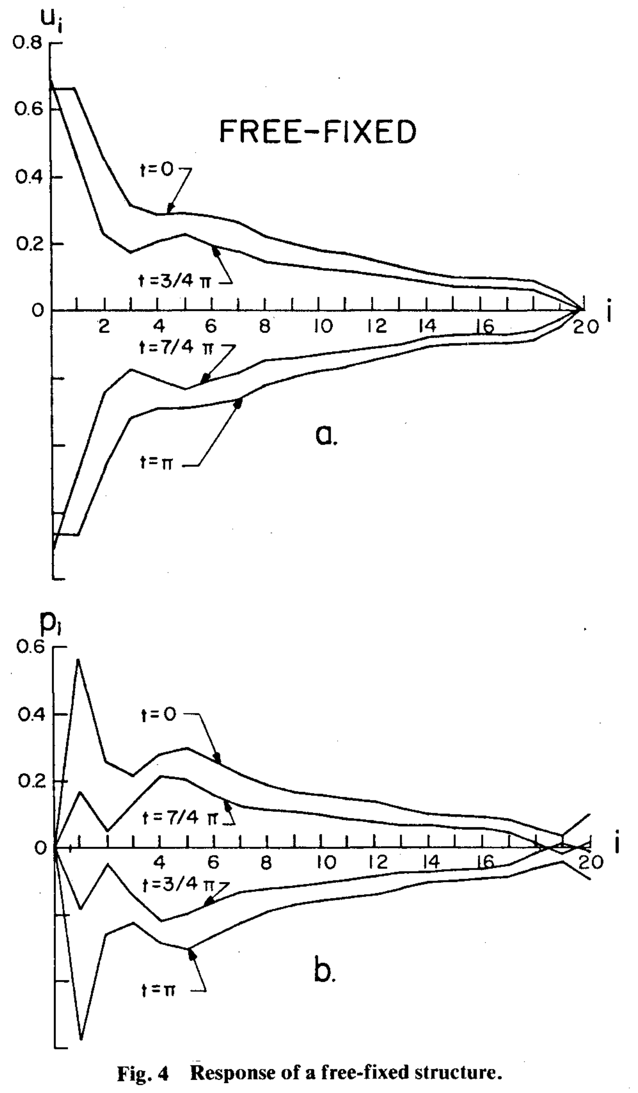


**Fig. 4 Response of a free-fixed structure.**

having the roots

$$z_1=\alpha+\sqrt{\alpha^2-1}\ ,\quad z_2=\alpha-\sqrt{\alpha^2-1} \tag{50}$$

and it is easy to verify that $z_2=1/z_1$. Using Eq. (20), we can write

$$X_1=\frac{1}{z_1-z_2}\begin{bmatrix} z_1-\alpha & -1/\beta \\ \beta(1-\alpha^2) & z_1-\alpha\end{bmatrix} =\frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{\beta\sqrt{\alpha^2-1}} \\ -\beta\sqrt{\alpha^2-1} & 1\end{bmatrix}$$
$$X_2=\frac{1}{z_2-z_1}\begin{bmatrix} z_2-\alpha & -1/\beta \\ \beta(1-\alpha^2) & z_2-\alpha\end{bmatrix} =\frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\beta\sqrt{\alpha^2-1}} \\ \beta\sqrt{\alpha^2-1} & 1\end{bmatrix} \tag{51}$$

Inserting Eqs. (51) into Eq. (22), we obtain the fundamental matrix

$$\Phi_i=\frac{1}{2}\begin{bmatrix} z_1^i+z_2^i & -\frac{1}{\beta\sqrt{\alpha^2-1}}(z_1^i-z_2^i) \\ -\beta\sqrt{\alpha^2-1}\ (z_1^i-z_2^i) & z_1^i+z_2^i\end{bmatrix} \tag{52}$$

Hence, using Eq. (12), we obtain a general solution for the response in the form

$$u_i=\frac{1}{2}(z_1^i+z_2^i)u_0-\frac{1}{2\beta\sqrt{\alpha^2-1}}\ (z_1^i-z_2^i)\,(p_0+f_0) -\frac{1}{2\beta\sqrt{\alpha^2-1}}\sum_{k=0}^{i-1}(z_1^{i-k-1}-z_2^{i-k-1})f_k^*$$
$$p_i=-\frac{\beta\sqrt{\alpha^2-1}}{2}\ (z_1^i-z_2^i)u_0+\frac{1}{2}(z_1^i+z_2^i)\,(p_0+f_0) +\sum_{k=0}^{i-1}(z_1^{i-k-1}+z_2^{i-k-1})f_k^*-f_{i,L} \tag{53}$$

At this point, it appears desirable to make the example more specific. Hence, let us assume that the system consists of 20 substructures and consider two cases, one in which both ends are free and the other in which the left end is free and the right end is fixed. From Fig. 2b, we conclude that the external load has the form

$$f_i^*=e^{-0.1i} \tag{54}$$

Moreover, we choose the following values for the system parameters

$$\omega=m=c=k=1 \tag{55}$$

From Eqs. (46), it follows that

$$\alpha=\tfrac{1}{2}(1+j),\qquad \beta=1+j \tag{56}$$

so that, from Eqs. (50), we obtain

$$z_1=0.74293+1.52908\ j$$
$$z_2=0.25707-0.52908\ j \tag{57}$$

Considering first the *free-free* case, we can write

$$p_0+f_0=0+e^0=1,\qquad p_{20}=0 \tag{58}$$

Letting $i=0$ in the second of Eqs. (53), we can solve for $u_0$ to obtain

$$u_0=\frac{2}{\beta\sqrt{\alpha^2-1}(z_1^{20}-z_2^{20})}\,[\tfrac{1}{2}(z_1^{20}+z_2^{20}) +\sum_{k=0}^{19}(z_1^{19-k}+z_2^{19-k})e^{-0.1k}-e^{-1.9}] \tag{59}$$

The sytem response is obtained by inserting Eqs. (56-59) into Eqs. (53). The actual evaluation is carried out most conveniently by a digital computer. The results are displayed in Figs. 3a and 3b.

The response for the *free-fixed* case is also given by Eqs. (53), but the end conditions are different. Indeed, in this case

$$p_0+f_0=1,\qquad u_{20}=0 \tag{60}$$

so that, from the first of Eqs. (53), we obtain

$$u_0=\frac{1}{\beta\sqrt{\alpha^2-1}(z_1^{20}+z_2^{20})}\,[z_1^{20}-z_2^{20} +\sum_{k=0}^{19}(z_1^{19-k}-z_2^{19-k})e^{-0.1k}] \tag{61}$$

Hence, using Eqs. (53) in conjunction with the end conditions (60-61), we obtain the response plotted in Figs. 4a and 4b.

Comparing the results obtained for the two cases, we observe that the response in the left part of the system is not affected very much by the end conditions at the right end, which, of course, can be attributed to the relatively large damping. This fact does permit us to draw the conclusion, however, that many periodic structures with a large number of elements can be regarded as being semi-infinite. In this case, significant computational saving can be obtained by discarding the part of the solution corresponding to roots of magnitude larger than unity.

## Summary and Conclusions

This paper presents an efficient method for the solution of matrix difference equations arising in connection with the response of periodic structures. The method is capable of deriving the response not only to end conditions but also to external excitations applied throughout the entire structure. The procedure is based on the Leverrier algorithm, and it requires only the eigenvalues of the transfer matrix of a single substructure. The algorithm is modified to account for the special dynamic properties of transfer matrices, thus achieving additional computational efficiency. No assumption has been made concerning the symmetry of the substructures. Indeed, added computational saving can be obtained for symmetric substructures. This subject will be discussed in a future paper.

As an illustrative example, a periodic structure with its substructures consisting of single-degree-of-freedom damped systems and subjected to external excitation has been solved. Note that none of the references cited gives explicit expressions for the response of periodic structures subjected to external distributed excitation. The example shows that for heavy damping the response of one end of the structure is virtually unaffected by the motion of the other end. Hence, under certain circumstances a finite structure can be treated as semi-infinite. The illustrative example, although demonstrating how the method works, does not nearly demonstrate the possibilities of the method. Indeed, the real power of the method becomes more evident as the number of degrees of freedom of the substructure increases.

The method holds great promise and should find its application in a large variety of physical systems which can be described by difference equations with constant coefficients.

## Appendix: Reduction of the Order of the Characteristic Equation

Let us write the characteristic polynomial, Eq. (41), in the form

$$\sum_{\ell=0}^{2n} \theta_\ell \lambda^{2n-\ell}=0 \tag{A1}$$

where we recall that $\theta_0=\theta_{2n}=1$. Dividing Eq. (A1) through by $\lambda^n$ and making use of Eqs. (34), we obtain

$$\sum_{\ell=1}^{n} \theta_{n-\ell}(\lambda^\ell+\lambda^{-\ell})+\theta_n=0 \tag{A2}$$

Introducing the notation

$$\lambda_0=1, \qquad \lambda_\ell=\lambda^\ell+\lambda^{-\ell}, \qquad l=1,2,\ldots,n \tag{A3}$$

Eq. (A2) can be written in the form of the vector product

$$\boldsymbol{\theta}^T\boldsymbol{\lambda}=0 \tag{A4}$$

where

$$\boldsymbol{\theta}=[\theta_n\ \theta_{n-1}\ldots\theta_0]^T, \quad \boldsymbol{\lambda}=(\lambda_0\lambda_1\ldots\lambda_n]^T \tag{A5}$$

Next, let

$$\gamma=\lambda+(1/\lambda) \tag{A6}$$

from which it follows that

$$\gamma^k=\sum_{\ell=0}^{k}\binom{k}{\ell}\lambda^{k-2\ell} \tag{A7}$$

where

$$\binom{k}{\ell}=\frac{k!}{\ell!(k-\ell)!} \tag{A8}$$

At this point, it will prove convenient to introduce a matrix $B=[b_{k\ell}]$ $(k,\ell=0,1,\ldots,n)$ defined as follows:

$$b_{k\ell}=\binom{k}{\frac{k-\ell}{2}} \quad \text{if } k \text{ and } \ell \text{ are both even or both odd}$$

$b_{k\ell}=0$ otherwise

Then, Eq. (A7) can be written in the form

$$\boldsymbol{\gamma}=B\boldsymbol{\lambda} \tag{A9}$$

where $\boldsymbol{\gamma}\neq[\gamma^0\gamma^1\ldots\gamma^n]^T$. Introducing Eq. (A9) into Eq. (A4), we obtain simply

$$\boldsymbol{\theta}^TB^{-1}\boldsymbol{\gamma}=0 \tag{A10}$$

From the definition of $B$, we conclude that the matrix is a unit lower triangular matrix, so that det $B=1$. It follows that $B^{-1}=\text{Adj } B$ is also a unit lower triangular matrix. When the notation

$$D=\text{Adj } B \tag{A11}$$

is introduced, where $D=[d_{k\ell}]$, Eq. (A10) becomes

$$\boldsymbol{\theta}^TD\boldsymbol{\gamma}=0 \tag{A12}$$

Moreover, it will prove convenient to introduce the vector

$$\boldsymbol{\delta}=D^T\boldsymbol{\theta} \tag{A13}$$

where $\boldsymbol{\delta}=[\delta_0\ \delta_1\ldots\delta_n]^T$. Equation (A13) has the index form

$$\delta_k=\sum_{\ell=0}^{n} d_{\ell k}\theta_{n-\ell} \tag{A14}$$

and because $d_{\ell k}=0$ for $\ell<k$, we have

$$\delta_k=\sum_{\ell=k}^{n} d_{\ell k}\theta_{n-\ell} \tag{A15}$$

On the other hand, for $\ell>k$ it can be easily verified that

$$d_{\ell k}=\det\begin{bmatrix} b_{k+1,k} & 1 & \cdots & 0 & 0 \\ b_{k+2,k} & b_{k+2,k+1} & \cdots & 0 & 0 \\ \hline b_{\ell-1,k} & b_{\ell-1,k+1} & \cdots & b_{\ell-1,\ell-2} & 1 \\ b_{\ell k} & b_{\ell,k+1} & \cdots & b_{\ell,\ell-2} & b_{\ell,\ell-1} \end{bmatrix} \tag{A16}$$

From this determinant it follows that $d_{\ell k}=0$ if $\ell+k$ is an odd number and

$$d_{\ell k}=\det\begin{bmatrix} b_{k+2,k} & 1 & \cdots & 0 & 0 \\ b_{k+4,k} & b_{k+4,k+2} & \cdots & 0 & 0 \\ \hline b_{\ell-2,k} & b_{\ell-2,k+2} & \cdots & b_{\ell-2,\ell-4} & 1 \\ b_{\ell k} & b_{\ell,k+2} & \cdots & b_{\ell,\ell-4} & b_{\ell,\ell-2} \end{bmatrix} \tag{A17}$$

if $\ell+k$ is an even number. This permits us to construct the following recurrence formula

$$d_{\ell k}=-\sum_{s=k,k+2,\ldots}^{\ell-2} b_{\ell s}d_{sk} \tag{A18}$$

in which it is recalled that

$$b_{\ell s}=\binom{\ell}{\frac{\ell-s}{2}}$$

In view of the above, Eq. (A15) reduces to

$$\delta_k=\sum_{\ell=k,k+2,\ldots}^{n} d_{\ell k}\theta_{n-\ell} \tag{A19}$$

Finally, considering Eqs. (A12) and (A13), the *reduced characteristic equation* can be written in the form

$$\sum_{k=0}^{n}\delta_k\gamma^k=0 \tag{A20}$$

where $\delta_k$ is given by Eq. (A19).

Equation (A20) yields $n$ roots $\gamma_1, \gamma_2,\ldots,\gamma_n$. Introducing these roots into Eq. (A6), we can obtain the $2n$ eigenvalues of $A$, i.e., $\lambda_1,\lambda_2,\ldots,\lambda_{2n}$.

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